Math 451: Introduction to General Topology
Lecture 25

Sequential compactness and products.

Theorem. A ctbl product of sequentially compact spaces is sequentially compact.

Proof (Arzela-Ascoli). We give an argument for ∞ products, which also proves it for finite products along the way. Let $X = \prod_{i \in \mathbb{N}} X_i$, where X_i is sequentially compact. Let $(f_n) \subseteq X$ and we need to find a convergent subsequence in the product top, i.e. a subsequence which converges pointwise at every $i \in \mathbb{N}$. Since X_0 is seq. comp. $\exists$ subsequence $(f_{n_{0k}})_{k \in \mathbb{N}}$ s.t. $(f_{n_{0k}}(0))_{k \in \mathbb{N}}$ converges in X_0 to some value $f(0) \in X_0$. Since X_1 is sequentially comp., $\exists$ further subsequence $(f_{n_{1k}})_{k \in \mathbb{N}}$ of $(f_{n_{0k}})_{k \in \mathbb{N}}$ s.t. $\lim_{k \to \infty} f_{n_{1k}}(1)$ exists and equal to some $f(1) \in X_1$. Continuing in this fashion (by induction), we obtain for each $i \in \mathbb{N}$ a sequence $(f_{n_{ik}})_{k \in \mathbb{N}}$ s.t. it is a subsequence of $(f_{n_{(i-1)k}})_{k \in \mathbb{N}}$ and $\lim_{k \to \infty} f_{n_{ik}}(i)$ exists and is equal to some $f(i) \in X_i$.

$$
\left.
\begin{array}{lllllll}
i = 0: & \boxed{f_{n_{00}}} & f_{n_{01}} & f_{n_{02}} & f_{n_{03}} & \dots & \to f(0) \text{ at } 0 \\
i = 1: & f_{n_{10}} & \boxed{f_{n_{11}}} & f_{n_{12}} & f_{n_{13}} & \dots & \to f(1) \text{ at } 1 \\
i = 2: & f_{n_{20}} & f_{n_{21}} & \boxed{f_{n_{22}}} & f_{n_{23}} & \dots & \to f(2) \text{ at } 2 \\
i = 3: & f_{n_{30}} & f_{n_{31}} & f_{n_{32}} & \boxed{f_{n_{33}}} & \dots & \to f(3) \text{ at } 3 \\
\vdots & & \vdots & & & \ddots &
\end{array}
\right\} \text{The Arzela-Ascoli diagonalization argument}
$$

Then the diagonal sequence $(f_{n_{kk}})_{k \in \mathbb{N}}$ is a subsequence of each $(f_{n_{ik}})_{k \in \mathbb{N}}$ after throwing out the first i member, which doesn't affect the limit (since limit is a tail property). Therefore for each $i \in \mathbb{N}$, $(f_{n_{kk}}(i))_{k \in \mathbb{N}}$ converges in X_i to $f(i)$. Thus $(f_{n_{kk}})_{k \in \mathbb{N}}$ converges in the product top to $f \in X$. $\square$

Remark. Using this one can characterize compact subsets of $C(X)$ where X is a separable metric space. This characterization is known as the Arzelà–Ascoli theorem. The statement and its proof will be outlined as practice for the final exam.

Baire category notions and theorems.

Let X be a top. space.

Def. A set $B \subseteq X$ is said to be dense in an open set $U \subseteq X$ if $B \cap U$ is a dense subset of U (in the relative top), i.e. B has a representative in each nonempty open $V \subseteq U$.

Obs. For $B \subseteq X$ and open $U \subseteq X$, B is dense in U $\iff$ $\overline{B} \supseteq U$.

Def. Call a set $B \subseteq X$ nowhere dense (n.d.) if B is not dense in any nonempty open set, i.e. each nonempty open $U \subseteq X$ admits a further nonempty open $V \subseteq U$ disjoint from B.

Obs. For $B \subseteq X$, TFAE:

(1) B is n.d.

(2) $\overline{B}$ is n.d.

(3) $\operatorname{int}(\overline{B}) = \emptyset$.

(4) B^c contains a dense open set.

Proof. (1) $\iff$ (3) $\iff$ (2) is by the observation above, while (3) $\iff$ (4) are just by taking complements. $\square$

Upgrade for n.d. Let $B \subseteq X$.

(a) If B is nowhere dense then it is contained in a closed n.d. set (namely, $\overline{B}$).

(a^c) If B is co-n.d. (i.e. B is a complement of a n.d. set), then B contains an open dense set (which is hence also co-n.d.).

Proof. This is just a rephrasing the above equivalences (1) $\iff$ (2) and (1) $\iff$ (4). $\square$

Examples. (a) In $\mathbb{R}$ or any other perfect metric space (i.e. no isolated points), singletons are nowhere dense because they are closed and have empty interior because they are not isolated.

(b) $\mathbb{Z} \subseteq \mathbb{R}$ is n.d. because it's closed and has empty interior.

(c) $\{\frac{1}{n} : n \in \mathbb{N}^+\} \subseteq \mathbb{R}$ is nowhere dense because its closure is $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^+\}$ has empty interior.

(d) The Cantor set $C \subseteq [0,1]$ is n.d. because it's closed and has empty interior.

(e) Let $k < \ell$ and $\ell \in \mathbb{N} \cup \{\mathbb{N}\}$. Then $k^{\mathbb{N}} \subseteq \ell^{\mathbb{N}}$ is n.d. because it's closed and has empty interior. In particular, $2^{\mathbb{N}}$ is n.d. in $3^{\mathbb{N}}$ and $k^{\mathbb{N}}$ is n.d. in $\mathbb{N}^{\mathbb{N}}$.

Prop. N.d. sets form an *ideal*, i.e. they are closed downward under subsets and finite unions.

Proof. For finite unions it suffices to show for two, by induction. Let B_1, B_2 be n.d. Let $U \subseteq X$ be nonempty open. Then $\exists\, \emptyset \neq$ open $V \subseteq U$ s.t. $B_1 \cap V = \emptyset$. Furthermore, $\exists\, \emptyset \neq$ open $W \subseteq V$ s.t. $B_2 \cap W = \emptyset$, so $(B_1 \cup B_2) \cap W = \emptyset$. $\square$

In analysis and topology, we take limits, so being closed under finite unions is not a good enough notion of smallness for us. We'd like being closed under ctbl unions.

Example. $\mathbb{Q} \subseteq \mathbb{R}$ is dense so it's not n.d. but $\mathbb{Q} = \bigcup_{n \in \mathbb{N}} \{q_n\}$ where $\mathbb{Q} = \{q_n\}_{n \in \mathbb{N}}$ and each singleton is n.d., so $\mathbb{Q}$ is a ctbl union of n.d. sets but it itself n.d.

Def. Call a set *meagre* if it is a ctbl union of n.d. sets.

Obs. Meagre sets form a *σ-ideal*, i.e. are closed under subsets and ctbl unions.

Examples. (a) $\mathbb{Q}$ is meagre in $\mathbb{R}$, as shown above.
(b) $B := \bigcup_{k \geq 2} k^{\mathbb{N}}$ is meagre in the Baire space $\mathbb{N}^{\mathbb{N}}$.

Call a set $B \subseteq X$ **comeagre** if its complement is meagre.

Upgrade for meagre.
(a) Every meagre set is contained in a meagre F_σ set (ctbl union of closed).
(a^c) Every comeagre set contains a ctbl intersection of dense open sets (in part. G_δ).
Proof. This just follows from the upgrade for n.d. $\square$

Def. A top. space X is called **Baire** if every nonempty set is nonmeagre.

Caution. A Baire space is different from the Baire space $\mathbb{N}^{\mathbb{N}}$, although the Baire space is a Baire by the Baire category thm below.

Prop. For a top. space X, TFAE:
(1) X is Baire, i.e. no nonempty open set is meagre.
(2) Comeagre sets in X are dense.
(3) Ctbl intersections of dense open sets are dense.

Proof. (1) $\Rightarrow$ (2). If $D \subseteq X$ is comeagre then D^c is meagre, hence doesn't contain any nonempty open set, so D intersects every nonempty open set, thus is dense.
(2) $\Rightarrow$ (3). Ctbl intersections of open dense sets are comeagre because their complement is a ctbl union of closed sets with empty interior.
(3) $\Rightarrow$ (2). This the upgrade property (a^c) for meagre sets.
(2) $\Rightarrow$ (1). We show neg(1) $\Rightarrow$ neg(2). If a $\emptyset \neq$ open $U \subseteq X$ is meagre then U^c is comeagre and not dense because it doesn't intersect U. $\square$

Baire Category Thm. Complete metric spaces are Baire. Also (locally) compact Hausdorff spaces are Baire.

Proof. We prove for a complete metric space X, as the proof for compact Hausdorff spaces is similar.

We need to show that ctbl intersections of dense open sets are dense. Let $U_n \subseteq X$ be dense open. To show that $\bigcap_{n \in \mathbb{N}} U_n$ is dense in X, fix a $\emptyset \neq$ open $V_0 \subseteq X$ and show $\bigcap_{n \in \mathbb{N}} U_n \cap V_0 \neq \emptyset$. Recall that metric spaces are normal. We play a two-player game, where the players alternate playing nonempty open sets, each contained in the previous one played. Player 2 wins if the intersection of all the played open sets is nonempty; and Player 1 wins otherwise. Player 1 starts with $U_0 \cap V_0$, which is nonempty because U_0 is dense. Then Player 2 uses the normality of X to play a nonempty open V_1 so that $\overline{V_1} \subseteq U_0 \cap V_0$ and $\operatorname{diam} \overline{V_1} < \frac{1}{1}$. Then Player 1 plays $U_1 \cap V_1$, which is nonempty because U_1 is dense. Then Player 2 plays a $\emptyset \neq$ open V_2 s.t. $\overline{V_2} \subseteq U_1 \cap V_1$ and $\operatorname{diam} V_2 < \frac{1}{2}$. And so on...

$$\begin{array}{lccccccc}
\underline{\text{Player 1:}} & U_0 \cap V_0 & & U_1 \cap V_1 & & U_2 \cap V_2 & & \\
 & \rotatebox{-45}{$\supseteq$} & \rotatebox{45}{$\supseteq$} & \rotatebox{-45}{$\supseteq$} & \rotatebox{45}{$\supseteq$} & \rotatebox{-45}{$\supseteq$} & & \\
\underline{\text{Player 2:}} & & V_1 & & V_2 & & V_3 & \cdots
\end{array}$$

Thus we obtain a decreasing sequence of nonempty open sets $U_0 \cap V_0 \supseteq V_1 \supseteq U_1 \cap V_1 \supseteq V_2 \supseteq \ldots$ s.t. $\overline{V_{n+1}} \subseteq U_n \cap V_n$ and $\operatorname{diam} V_n \leq \frac{1}{n}$ $\forall n \in \mathbb{N}^+$. Thus, $\bigcap_{n \geq 1} U_n \cap V_n = \bigcap_{n \geq 1} \overline{V_n} = \bigcap_{n \geq 1} V_n$. But by the completeness of X, $\bigcap_{n \geq 1} \overline{V_n} \neq \emptyset$, so $\bigcap_{n \geq 1} (U_n \cap V_n) \neq \emptyset$, in particular $V_0 \cap \bigcap_{n \geq 1} U_n \neq \emptyset$, as desired. $\square$

Applications. One can show that in $C([0,1])$, the nowhere differentiable functions form a comeager set, hence $\exists$ a nowhere differentiable function. We will show that nowhere monotone functions also form a comeager set. To see this, fix any two rationals $0 \leq a < b \leq 1$, let $B_{a,b}$ be the set of $f \in C([0,1])$ s.t. f is monotone on $[a,b]$. One verifies this set is closed in the uniform metric and has $\emptyset$ interior because we can create a small zigzag between a and b on the graph of the function. Thus, $B := \bigcup_{0 \leq a < b \leq 1} B_{a,b}$ is meagre, but every somewhere monotone function is monotone in $[a,b]$, so B is the set of all somewhere monotone function, so the nowhere monotone functions form a comeagre (hence nonempty) set.